

# The behaviour of solutions of the Gaussian curvature equation near an isolated boundary point

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**Abstract.** A classical result of Nitsche [21] about the behaviour of the solutions to the Liouville equation  $\Delta u = 4e^{2u}$  near isolated singularities is generalized to solutions of the Gaussian curvature equation  $\Delta u = -\kappa(z)e^{2u}$  where  $\kappa$  is a negative Hölder continuous function. As an application a higher-order version of the Yau–Ahlfors–Schwarz lemma for complete conformal Riemannian metrics is obtained.

## 1 Introduction

In [21] Nitsche gave a detailed description of the behaviour of the real-valued solutions to the Liouville equation

$$\Delta u = 4e^{2u} \quad (1.1)$$

on plane domains near their isolated singularities. The first purpose of the present paper is to extend Nitsche's results to the solutions of the more general Gaussian curvature equation

$$\Delta u = -\kappa(z)e^{2u} \quad (1.2)$$

with strictly negative Hölder continuous functions  $\kappa(z)$ . It suffices to consider this PDE on the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  in the complex plane  $\mathbb{C}$ . We use the notation

$$M_u(r) := \sup_{|z|=r} u(z)$$

for real-valued functions  $u$  defined in a punctured neighborhood of  $z = 0$  and call

$$\alpha(u) := \lim_{r \searrow 0} \frac{M_u(r)}{\log(1/r)}$$

the order of  $u$  if this limit exists.

### Theorem 1.1

Let  $\kappa : \mathbb{D} \rightarrow \mathbb{R}$  be a locally Hölder continuous function with  $\kappa(0) < 0$ . If  $u : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}$  is a  $C^2$ -solution to  $\Delta u = -\kappa(z)e^{2u}$  in  $\mathbb{D} \setminus \{0\}$ , then  $u$  has order  $\alpha \in (-\infty, 1]$  and

$$u(z) = -\alpha \log |z| + v(z), \quad \text{if } \alpha < 1, \quad (1.3)$$

$$u(z) = -\log |z| - \log \log(1/|z|) + w(z), \quad \text{if } \alpha = 1, \quad (1.4)$$

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where the remainder functions  $v$  and  $w$  are continuous in  $\mathbb{D}$ . Moreover, the first partial derivatives with respect to  $z$  and  $\bar{z}$ ,

$$v_z(z), v_{\bar{z}}(z) \text{ are continuous at } z = 0 \quad \text{if } \alpha < 1/2;$$

and

$$\begin{aligned} v_z(z), v_{\bar{z}}(z) &= O(1) & \text{if } \alpha = 1/2, \\ v_z(z), v_{\bar{z}}(z) &= O(|z|^{1-2\alpha}) & \text{if } 1/2 < \alpha < 1, \\ w_z(z), w_{\bar{z}}(z) &= O(|z|^{-1} (\log(1/|z|))^{-2}) & \text{if } \alpha = 1, \end{aligned}$$

when  $z$  approaches  $z = 0$ . In addition, the second partial derivatives,

$$v_{zz}(z), v_{z\bar{z}}(z) \text{ and } v_{\bar{z}\bar{z}}(z) \text{ are continuous at } z = 0 \quad \text{if } \alpha \leq 0;$$

and

$$\begin{aligned} v_{zz}(z), v_{z\bar{z}}(z), v_{\bar{z}\bar{z}}(z) &= O(|z|^{-2\alpha}) & \text{if } 0 < \alpha < 1, \\ w_{zz}(z), w_{z\bar{z}}(z), w_{\bar{z}\bar{z}}(z) &= O(|z|^{-2} (\log(1/|z|))^{-2}) & \text{if } \alpha = 1, \end{aligned}$$

when  $z$  tends to  $z = 0$ .

Theorem 1.1 merits some comment. Firstly, the special case  $\kappa(z) \equiv -4$  of Theorem 1.1 is Nitsche's theorem; see [21, Satz 1].<sup>1</sup> Nitsche's proof is based on an ingenious application of Liouville's classical representation formula [17] for the solutions to (1.1). Roughly speaking, Liouville's result says that in any disk  $D \subseteq \mathbb{D} \setminus \{0\}$  every solution  $u$  to (1.1) can be written as

$$u(z) = \log \frac{|f'(z)|}{1 - |f(z)|^2},$$

where  $f$  is a holomorphic function in  $D$ . Now a careful and clever study of the analytic continuation of  $f$  along a path surrounding the singularity  $z = 0$  enables Nitsche to prove Theorem 1.1 for  $\kappa(z) \equiv -4$ . The same argument was later used by Yamada [32] and also by Chou and Wan [7, 8] who were apparently unaware of Nitsche's paper. Clearly, Nitsche's method cannot be applied to prove Theorem 1.1 for non-constant functions  $\kappa(z)$  as there is no representation formula of Liouville-type in this case.

Secondly, the motivation for extending Nitsche's result mainly comes from the geometric interpretation of the PDE (1.2): every solution  $u$  to (1.2) gives rise to a conformal Riemannian metric  $e^{u(z)} |dz|$  with Gaussian curvature  $\kappa(z)$  and vice versa; see Paragraph 2.1 below. Thus passing from Liouville's equation (1.1) to the more general equation (1.2) amounts to passing from constantly curved conformal Riemannian metrics to metrics with variable curvature.

We note that the constant curvature case is intimately connected with the uniformization problem for Riemann surfaces and the classical Schwarz–Picard problem. Its study was pursued by Schwarz [29], Picard [22, 23], Poincaré [24], Bieberbach [3, 4], Heins [12] and many others. It has only recently led to a complete proof of the uniformization theorem for Riemann surfaces by Mazzeo and Taylor [18] solely based on curvature considerations. The Schwarz–Picard problem is the problem of investigating the solutions to  $\Delta u = 4e^{2u}$  with prescribed singularities on a compact Riemann surface. The solution to the existence–and–uniqueness part of the Schwarz–Picard problem is due to Heins [12], while the sharp growth

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<sup>1</sup>Nitsche considers the PDE  $\Delta U = e^U$ , which is obtained from  $\Delta u = 4e^{2u}$  using the transformation  $U(z) = 2u(z) + \log 8$ ; cf. also Remark 2.2 below.

and regularity properties of the corresponding solutions at the singularities are described by Nitsche's theorem.

The more general and difficult case of variable curvature is strongly related to the Berger–Nirenberg problem in differential geometry (see Aviles & McOwen [2], Troyanov [30], Hulin & Troyanov [14], and Chang [6] as some of the many references). In particular, in analogy to the Schwarz–Picard problem the solutions to the equation  $\Delta u = -\kappa(z)e^{2u}$  with prescribed singularities were studied in [30, 14, 19] and existence and uniqueness of solutions for strictly negative curvature functions were obtained. Theorem 1.1 supplements these results by extending Nitsche's theorem to the variable curvature case and thus establishing the corresponding sharp growth and regularity properties of the solutions near their singularities.

The basic ingredients we employ in the present paper to carry over all of Nitsche's results for the constant curvature case to the variable curvature case are a generalized maximum principle for the Gaussian curvature equation (*cf.* Theorem 2.4 below), which allows an application of the method of sub- and supersolutions, and potential-theoretic tools. Our approach reveals precisely how the growth and regularity of the remainder functions  $v$  and  $w$  at  $z = 0$  depend on the regularity of the curvature function  $\kappa$ . It also leads to a number of refinements of Theorem 1.1 with weaker assumptions. These refinements will be discussed in detail in Section 3. For instance, we shall see that if  $u(z)$  is of the form (1.4), then the function  $w$  is continuous, when  $\kappa(z)$  is only assumed to be continuous at  $z = 0$  (*cf.* Theorem 3.4 and Example 3.3).

As indicated above Theorem 1.1 and its refinements (see Section 3) give precise information about the behaviour of regular<sup>2</sup> conformal Riemannian metrics and their first and second derivatives near isolated singularities. In order to state these information in more geometric terms we first recall the definition of two natural derivatives associated with regular conformal metrics. The connection (or Pre-Schwarzian or Christoffel symbol) of a regular conformal metric  $\lambda(z)|dz|$  on a plane domain  $\Omega \subset \mathbb{C}$  is defined by

$$\Gamma_\lambda(z) = 2 \frac{\partial \log \lambda(z)}{\partial z}$$

and the Schwarzian of  $\lambda(z)|dz|$  is given by

$$S_\lambda(z) = \frac{\partial \Gamma_\lambda(z)}{\partial z} - \frac{1}{2} \Gamma_\lambda(z)^2 = 2 \left[ \frac{\partial^2 \log \lambda(z)}{\partial z^2} - \left( \frac{\partial \log \lambda(z)}{\partial z} \right)^2 \right].$$

These differential quantities obey simple transformation laws under conformal change of coordinates; see [20] and [28].

### Theorem 1.2

*Let  $\lambda(z)|dz|$  be a regular conformal Riemannian metric on a domain  $\Omega \subset \mathbb{C}$  with an isolated boundary point at  $z = 0$ , and suppose that the curvature  $\kappa : \Omega \rightarrow \mathbb{R}$  has a Hölder continuous extension to  $\Omega \cup \{0\}$  such that  $\kappa(0) < 0$ . Then  $\log \lambda$  has order  $\alpha \in (-\infty, 1]$  and*

$$(a) \quad \lim_{z \rightarrow 0} (|z| \log(1/|z|)) \lambda(z) = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1/\sqrt{-\kappa(0)} & \text{if } \alpha = 1; \end{cases}$$

$$(b) \quad \lim_{z \rightarrow 0} z \Gamma_\lambda(z) = -\alpha;$$

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<sup>2</sup>We call a conformal metric  $\lambda(z)|dz|$  on a domain  $\Omega \subseteq \mathbb{C}$  regular, if its density  $\lambda$  is of class  $C^2$  in  $\Omega$ .

$$(c) \lim_{z \rightarrow 0} z^2 S_\lambda(z) = \alpha(2 - \alpha)/2.$$

**Remark 1.3**

Theorem 1.2 extends a result of D. Minda [20] who proved Theorem 1.2 for the special case that  $\lambda(z)|dz|$  is the hyperbolic metric in  $\Omega$  (with constant curvature  $-4$ ). He asked ([20, §4]) for an generalization of his results to metrics with *constant* negative curvature. Theorem 1.2 gives a complete answer to Minda's question even for the much more general case of metrics with *variable* negative curvature.

Another application of Theorem 1.1 deals with a higher-order version of the so-called Yau–Ahlfors–Schwarz lemma; see Yau [33] and Ahlfors [1]. In the complex one-dimensional case Yau's generalized Ahlfors–Schwarz lemma says that if  $\lambda_\Omega|dz|$  is the hyperbolic metric of a (hyperbolic) domain  $\Omega \subset \mathbb{C}$  and  $\lambda(z)|dz|$  is a *complete* regular conformal metric in  $\Omega$  with curvature  $\kappa(z) \geq -4$ , then  $\lambda(z) \geq \lambda_\Omega(z)$  for all  $z \in \Omega$ . Thus Yau's lemma derives a global estimate for conformal metrics from global assumptions. There are boundary versions of these results (*cf.* Bland [5], Troyanov [31], and Kraus, Roth & Ruscheweyh [16]). The following theorem gives precise *local* information about conformal metrics *and* its derivatives up to second order at isolated boundary points under *local* assumptions. We call a conformal metric  $\lambda(z)|dz|$  on a domain  $\Omega$  with an isolated boundary point at  $z = 0$  *locally complete* (at  $z = 0$ ) if

$$\lim_{\substack{z \rightarrow 0 \\ z \in \Omega}} d_\lambda(z_0, z) = +\infty$$

for some (and then every) point  $z_0 \in \Omega$ . Here  $d_\lambda$  denotes the distance function induced by  $\lambda(z)|dz|$ . A basic reference about locally complete metrics is the work of Huber [13]; see also [5] and [16].

**Theorem 1.4 (Higher-order Yau–Ahlfors–Schwarz Lemma)**

Let  $\Omega$  be a hyperbolic domain with an isolated boundary point at  $z = 0$  and let  $\lambda_\Omega(z)|dz|$  be the hyperbolic metric of  $\Omega$  (with curvature  $-4$ ). Let  $\lambda(z)|dz|$  be a locally complete conformal Riemannian metric on  $\Omega$  with Hölder continuous curvature  $\kappa : \Omega \cup \{0\} \rightarrow \mathbb{R}$  such that  $\kappa_\lambda(0) = -4$ . Then

$$(a) \lim_{z \rightarrow 0} \frac{\lambda(z)}{\lambda_\Omega(z)} = 1$$

$$(b) \lim_{z \rightarrow 0} \frac{\Gamma_\lambda(z)}{\Gamma_{\lambda_\Omega}(z)} = 1$$

$$(c) \lim_{z \rightarrow 0} \frac{S_\lambda(z)}{S_{\lambda_\Omega}(z)} = 1.$$

The paper is organized as follows. In Section 2 we begin with a discussion of some basic facts from conformal geometry (§2.1) and prove the extended maximum principle for the curvature equation (§2.2). Section 3 contains the statements and proofs of the main results of this work. It starts in Paragraph §3.1 with a proof of the representation formulas (1.3) and (1.4) of Theorem 1.1 under minimal hypotheses on the curvature function (Theorem 3.1) and a discussion, when the remainder functions  $v(z)$  and  $w(z)$  are continuous (see Theorem 3.4). We then establish the growth and regularity properties of the first derivatives of  $v(z)$  and  $w(z)$  in §3.2–§3.3 and of the second derivatives in §3.4 again under minimal assumptions on the curvature function. Theorems 1.2 and 1.4 are proved in §3.5. All of our results are

essentially sharp as will be illustrated with a number of examples. The paper ends with an appendix on the required (non-standard) tools from potential theory.

## 2 Preliminaries

As indicated above the proof of Theorem 1.1 uses a mixture of different methods from conformal geometry, subharmonic functions including a maximum-principle, and potential theory. In this preparatory section, we first recall some basic facts about conformal Riemannian metrics and show that Theorem 1.1 is best possible (§2.1). Paragraph 2.2 is devoted to an extended maximum principle for the Gaussian curvature equation which is the main technical tool for the proof of Theorem 1.1.

### 2.1 Conformal Riemannian metrics and the Gaussian curvature equation

Every positive upper semi-continuous function  $\lambda$  on a domain  $G \subset \mathbb{C}$  induces a conformal Riemannian metric  $\lambda(z) |dz|$  on  $G$ . The (Gaussian) curvature of a regular conformal Riemannian metric  $\lambda(z) |dz|$  is defined by

$$\kappa_\lambda(z) := -\frac{(\Delta \log \lambda)(z)}{\lambda(z)^2}, \quad \text{where } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad z = x + iy.$$

Thus, if  $\lambda(z) |dz|$  is a regular conformal Riemannian metric on a domain  $G$  with curvature  $\kappa_\lambda$ , then the function  $u := \log \lambda$  is a  $C^2$ -solution to the Gaussian curvature equation

$$\Delta u = -\kappa_\lambda(z) e^{2u}$$

in  $G$  and vice versa.

A basic property of curvature is its absolute conformal invariance. This means that the pull-back of a conformal Riemannian metric  $\lambda(z) |dz|$  on a domain  $D$ , defined by  $(f^*\lambda)(z) |dz| := \lambda(w) |dw|$ , where  $w = f(z)$  is a holomorphic map from a domain  $G$  to  $D$ , is a conformal Riemannian metric on  $G$  off the set of critical points of  $f$  with curvature

$$\kappa_{f^*\lambda}(z) = \kappa_\lambda(f(z)).$$

This conformal invariance provides a simple, but flexible tool to construct new conformal Riemannian metrics from old ones. For instance, the Poincaré metric

$$\frac{|dz|}{2|z| \log(1/|z|)}$$

on the punctured unit disk  $\mathbb{D} \setminus \{0\}$  has constant curvature  $-4$ . Pulling back this metric via the map  $f(z) = z/R$  for some  $R > 1$  gives another conformal Riemannian metric on  $\mathbb{D} \setminus \{0\}$  with curvature  $-4$ . The following examples have been constructed along these lines.

#### Example 2.1

*The function*

$$u_\alpha(z) = \begin{cases} -\alpha \log |z| + \log \left( (1-\alpha) \frac{4^{2-\alpha}}{2} \frac{|1+z| |2+z|^{-\alpha}}{4^{2(1-\alpha)} - |2z+z^2|^{2(1-\alpha)}} \right) & \text{if } \alpha \leq 0, \\ -\alpha \log |z| + \log \left( \frac{1-\alpha}{1-|z|^{2(1-\alpha)}} \right) & \text{if } 0 < \alpha < 1, \\ -\log |z| - \log \log \frac{1}{|z|} + \log \left( \frac{1}{2} \cdot \frac{\log(1/|z|)}{1 + \log(1/|z|)} \right) & \text{if } \alpha = 1. \end{cases}$$

is a  $C^2$ -solution to  $\Delta u = 4e^{2u}$  in  $\mathbb{D} \setminus \{0\}$ . Thus all cases  $\alpha \leq 1$  in Theorem 1.1 do occur. In the notation of Theorem 1.1 we obtain for the partial derivatives of the remainder functions  $v$  (if  $\alpha < 1$ ) and  $w$  (if  $\alpha = 1$ )

$$\begin{aligned} \lim_{z \rightarrow 0} v_z(z) &= \frac{1}{2} - \frac{\alpha}{4} && \text{if } \alpha \leq 0, \\ v_z(z) &= \frac{1 - \alpha}{1 - |z|^{2(1-\alpha)}} \frac{\bar{z}}{|z|^{2\alpha}} && \text{if } 0 < \alpha < 1, \\ w_z(z) &= -\frac{1}{2z (\log(1/|z|))^2} \left( \frac{\log(1/|z|)}{1 + \log(1/|z|)} \right) && \text{if } \alpha = 1, \end{aligned}$$

and

$$\begin{aligned} \lim_{z \rightarrow 0} v_{zz}(z) &= -\frac{1}{2} + \frac{\alpha}{8} && \text{if } \alpha \leq 0, \\ v_{zz}(z) &= (1 - \alpha) \frac{\bar{z}}{z} \frac{|z|^{2(1-\alpha)} - \alpha}{(1 - |z|^{2(1-\alpha)})^2} \frac{1}{|z|^{2\alpha}} && \text{if } 0 < \alpha < 1, \\ w_{zz}(z) &= -\frac{1}{4z^2 (\log(1/|z|))^2} \frac{\frac{1}{(\log(1/|z|))^2} - 2}{\left(1 + \frac{1}{\log(1/|z|)}\right)^2} && \text{if } \alpha = 1. \end{aligned}$$

In particular, all the statements of Theorem 1.1 are best possible even for the constant curvature case.

### Remark 2.2

We note at this point that Nitsche [21, Satz 1] erroneously asserts that the first partial derivatives  $v_z, v_{\bar{z}} = O(|z|^{1-2\alpha})$  and the second partial derivatives of  $v$  are in  $O(|z|^{-2\alpha})$  for all  $\alpha < 1$ . This, however, as the above examples show, is only true for  $1/2 < \alpha < 1$  and  $0 < \alpha < 1$  respectively.

For completeness we also notice that in Theorem 1.1 the condition that  $\kappa(z)$  is bounded from above and below by negative constants at least close to  $z = 0$  cannot be dropped completely. For instance, the function

$$u(z) = \frac{1}{2} \log \left( \frac{1}{|z| \log(e/|z|)} \right) = -\frac{1}{2} \log |z| - \frac{1}{2} \log \log \frac{1}{|z|} + \frac{1}{2} \log \left( \frac{\log(1/|z|)}{1 + \log(1/|z|)} \right)$$

is a  $C^2$ -solution to the PDE

$$\Delta u = \frac{1}{|z|} \frac{1}{2 - 2 \log |z|} e^{2u}$$

in  $\mathbb{D} \setminus \{0\}$ . In this case  $\kappa(z) \rightarrow -\infty$  as  $z \rightarrow 0$ .

## 2.2 Subharmonic functions and an extended maximum principle

If the curvature function  $\kappa(z)$  is non-positive, then every solution to  $\Delta u = -\kappa(z) e^{2u}$  is obviously subharmonic. In order to exploit this property we need to make use of a number of facts about subharmonic functions.

For convenience we shall reserve the notation  $K_R := K_R(0)$  for the open disk with center 0 and radius  $R$ . Let  $u$  be a subharmonic function on the punctured disk  $K_R \setminus \{0\}$  with  $u \not\equiv -\infty$ . For  $0 < r < R$  let

$$M_u(r) := \sup_{|z|=r} u(z).$$

Then  $M_u(r)$  is a convex function of  $\log r$  (see [11, p. 67–68]), so the left and right derivatives of  $r \mapsto M_u(r)$  exist everywhere in  $0 < r < R$  and are equal outside a countable set. We denote the derivative by  $M'_u(r)$ . Also,  $rM'_u(r)$  is monotonically increasing and

$$\lim_{r \rightarrow 0} rM'_u(r) = - \lim_{r \rightarrow 0} \frac{M_u(r)}{\log(1/r)} \in [-\infty, +\infty) \quad (2.1)$$

exists (cf. [11, p. 67/68]). In particular, if

$$\lim_{r \rightarrow 0} \frac{M_u(r)}{\log(1/r)} = 0, \quad (2.2)$$

then  $M'_u(r) \geq 0$  for  $0 < r < R$ , so  $M_u(r)$  is monotonically increasing, and  $u$  is therefore bounded above on  $\overline{K_r} \setminus \{0\}$  for  $r < R$ . But then  $u$  has a subharmonic extension to  $K_R$ , when we put

$$u(0) := \limsup_{z \rightarrow 0} u(z) \in [-\infty, +\infty),$$

see [26, p. 48]. Thus (2.2) implies that  $z = 0$  is a removable singularity of the subharmonic function  $u$ . We will need the following variant of this simple fact:

### Lemma 2.3

Let  $u$  be an upper semi-continuous function and let  $v$  be a subharmonic function on  $K_R \setminus \{0\}$  with  $u, v \not\equiv -\infty$  such that

- (i)  $\lim_{r \rightarrow 0} \frac{M_u(r)}{\log(1/r)} = \lim_{r \rightarrow 0} \frac{M_v(r)}{\log(1/r)} < +\infty$ ; and
- (ii)  $u - v$  is a non-negative subharmonic function in  $K_R \setminus \{0\}$ .

Then  $u - v$  is subharmonic in  $z = 0$ .

### Proof.

Clearly,  $u = (u - v) + v$  is subharmonic in  $K_R \setminus \{0\}$ . Denoting the common value of the limits in condition (i) by  $\alpha \in \mathbb{R}$ , we observe that

$$\lim_{r \rightarrow 0} \frac{M_{u_\alpha}(r)}{\log(1/r)} = \lim_{r \rightarrow 0} \frac{M_{v_\alpha}(r)}{\log(1/r)} = 0$$

for  $u_\alpha(z) = u(z) - \alpha \log(1/|z|)$  and  $v_\alpha(z) = v(z) - \alpha \log(1/|z|)$ . Clearly,  $u_\alpha$  and  $v_\alpha$  are subharmonic in  $K_R \setminus \{0\}$ . By the discussion preceding Lemma 2.3, we thus obtain that  $u_\alpha$  and  $v_\alpha$  are subharmonic on the entire disk  $K_R$ , so by [27, p. 48 and p. 78]

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{1}{2\pi} \int_0^{2\pi} \frac{u(re^{it}) - v(re^{it})}{\log(1/r)} dt &= \lim_{r \rightarrow 0} \frac{1}{2\pi} \int_0^{2\pi} \frac{u_\alpha(re^{it})}{\log(1/r)} dt - \lim_{r \rightarrow 0} \frac{1}{2\pi} \int_0^{2\pi} \frac{v_\alpha(re^{it})}{\log(1/r)} dt \\ &= \lim_{r \rightarrow 0} \frac{M_{u_\alpha}(r)}{\log(1/r)} - \lim_{r \rightarrow 0} \frac{M_{v_\alpha}(r)}{\log(1/r)} = 0. \end{aligned}$$

Hence a result of BreLOT and Saks ([26, p. 49]) guarantees that the non-negative function  $u - v$  is subharmonic at  $z = 0$ , as required.  $\blacksquare$

Lemma 2.3 is now used to derive a maximum principle for subharmonic sub- and supersolutions to the PDE  $\Delta u = -\kappa(z) e^{2u}$  on the punctured disk  $K_R \setminus \{0\}$ . The only information we need at  $z = 0$  is encoded in the value of the limit (2.1). This is indeed a very useful gain in flexibility and as we shall see later the key to Theorem 1.1. Recall that a  $C^2$ -function  $u$  is a subsolution to  $\Delta u = -\kappa(z) e^{2u}$  if  $\Delta u \geq -\kappa(z) e^{2u}$  and a supersolution to  $\Delta u = -\kappa(z) e^{2u}$  if  $\Delta u \leq -\kappa(z) e^{2u}$ .

**Theorem 2.4 (Extended maximum principle for the curvature equation)**

Let  $\kappa : K_R \setminus \{0\} \rightarrow \mathbb{R}$  be a non-positive function and let  $u_1, u_2 : K_R \setminus \{0\} \rightarrow \mathbb{R}$ , where

- (i)  $u_2$  is a subharmonic supersolution to  $\Delta u = -\kappa(z) e^{2u}$  in  $K_R \setminus \{0\}$ ;
- (ii)  $u_1$  is a subsolution to  $\Delta u = -\kappa(z) e^{2u}$  in  $K_R \setminus \{0\}$ ;
- (iii)  $\limsup_{z \rightarrow \xi} u_1(z) \leq \liminf_{z \rightarrow \xi} u_2(z)$  for every  $\xi \in \partial K_R$ ; and
- (iv)  $\lim_{r \rightarrow 0} \frac{M_{u_1}(r)}{\log(1/r)} \leq \lim_{r \rightarrow 0} \frac{M_{u_2}(r)}{\log(1/r)} < +\infty$ .

Then  $u_1 \leq u_2$  in  $K_R \setminus \{0\}$ .

**Remark 2.5**

Theorem 2.4 can be easily extended to much more general nonlinear (twodimensional) PDEs such as the class of PDEs discussed in [15, §2.3] and [10, §10.1].

**Proof.**

In order to prove Theorem 2.4 we apply Lemma 2.3. At first we define on  $K_R \setminus \{0\}$  the subharmonic function

$$w_1(z) = \max\{u_1(z), u_2(z)\}.$$

Then the function

$$w_2(z) := w_1(z) - u_2(z)$$

is non-negative and by condition (i) and (ii) subharmonic on  $K_R \setminus \{0\}$ . In fact, if  $w_2(z_0) > 0$  at some point  $z_0 \in K_R \setminus \{0\}$ , then  $w_2(z) = u_1(z) - u_2(z) > 0$  in a neighborhood of  $z_0$ . Thus

$$\Delta w_2(z) = \Delta u_1(z) - \Delta u_2(z) \geq -\kappa(z) \left( e^{2u_1(z)} - e^{2u_2(z)} \right) \geq 0$$

there, i.e.,  $w_2$  is subharmonic in this neighborhood. If  $w_2(z_0) = 0$  for some point  $z_0 \in K_R \setminus \{0\}$ , then  $w_2$  satisfies the submean inequality

$$w_2(z_0) = 0 \leq \frac{1}{2\pi} \int_0^{2\pi} w_2(z_0 + r e^{it}) dt$$

for all  $r > 0$  small enough. Hence  $w_2$  is subharmonic on  $K_R \setminus \{0\}$ .

Further, condition (iv) implies

$$\lim_{r \rightarrow 0} \frac{M_{w_1}(r)}{\log(1/r)} = \lim_{r \rightarrow 0} \frac{M_{u_2}(r)}{\log(1/r)} < +\infty$$



and  $w_2$  has therefore a subharmonic extension to  $K_R$  by Lemma 2.3. Thanks to the boundary behaviour of  $u_1$  and  $u_2$ , cf. assumption (iii), we have

$$\limsup_{z \rightarrow \xi} w_2(z) \leq 0 \quad \text{for every } \xi \in \partial K_R.$$

Applying the maximum principle for subharmonic functions to  $w_2$ , we deduce that  $w_2 \leq 0$  in  $K_R$  and so  $u_1 \leq u_2$  in  $K_R \setminus \{0\}$ .  $\blacksquare$

The following two examples illustrate that neither the assumption “ $u_2$  is subharmonic” in condition (i) nor the assumption  $\lim_{r \rightarrow 0} M_{u_2}(r)/\log(1/r) < +\infty$  in condition (iv) of Theorem 2.4 can be dropped.

**Example 2.6**

We pick  $\kappa \equiv 0$ , and choose  $u_1 \equiv 0$  and

$$u_2 : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}, \quad z \mapsto - \left( \frac{\operatorname{Re}(z)}{|z|} + 1 \right) \frac{1}{|z|^{3/2}} \log \frac{1}{|z|}.$$

It is easy to check that the function  $u_2$  is superharmonic in  $\mathbb{D} \setminus \{0\}$  and therefore a supersolution to  $\Delta u = -\kappa(z) e^{2u}$ . Obviously,  $u_1$  is a subsolution to  $\Delta u = -\kappa(z) e^{2u}$  in  $\mathbb{D}$  and  $u_1 \leq u_2$  on  $\partial \mathbb{D}$ . However  $u_1 \not\leq u_2$  in  $\mathbb{D} \setminus \{0\}$ , although

$$\lim_{r \rightarrow 0} \frac{M_{u_1}(r)}{\log(1/r)} = \lim_{r \rightarrow 0} \frac{M_{u_2}(r)}{\log(1/r)} = 0.$$

**Example 2.7**

Here we set  $\kappa \equiv -e^2$  and consider on  $\mathbb{D} \setminus \{0\}$  the harmonic function  $u_2(z) = \operatorname{Re}(z)/|z|^2$ , where

$$\inf_{|z|=1} u_2(z) = -1 \quad \text{and} \quad \lim_{r \rightarrow 0} \frac{M_{u_2}(r)}{\log(1/r)} = +\infty.$$

For the function  $u_1 : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}$  we choose

$$u_1(z) = \log \left( \frac{1}{e} \frac{1}{|z| \log(e/|z|)} \right).$$

Thus  $u_1$  is a solution to the “boundary value problem”

$$\Delta u = e^2 e^{2u} \quad \text{in } \mathbb{D} \setminus \{0\}, \quad u \equiv -1 \quad \text{on } \partial \mathbb{D}$$

which satisfies

$$\lim_{r \rightarrow 0} \frac{M_{u_1}(r)}{\log(1/r)} = 1.$$

Clearly,  $u_1 \not\leq u_2$  in  $\mathbb{D} \setminus \{0\}$ .

### 3 Main results and proofs

We are now prepared to prove Theorem 1.1. We shall see in §3.1 that the representation formulas (1.3) and (1.4) as well as the continuity properties of the remainder functions  $v$  and  $w$  follow from the extended maximum principle (Theorem 2.4) and the potential theory of Section 4 by constructing suitable sub- and supersolutions with *constant* curvature. In order to describe the precise behaviour of the derivatives of the remainder functions, however, one needs to find appropriate sub- and supersolutions with *variable* curvature; cf. §3.2–§3.4.

It is convenient to introduce the following notion.

**Definition 3.1**

A real-valued function  $\kappa$  on a set  $G \subseteq \mathbb{C}$  is called *strictly negative* on  $G$  if  $-a \leq \kappa(z) \leq -A$  in  $G$  for finite constants  $a > 0$  and  $A > 0$ . A function  $\kappa : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}$  is called *strictly negative* at  $z = 0$ , if  $\kappa$  is strictly negative in some punctured neighborhood of  $z = 0$ .

**3.1 Classification of the isolated singularities and continuity of the remainder functions**

We start with a discussion of the behaviour of the solutions to  $\Delta u = -\kappa(z) e^{2u}$  under the assumption that the curvature function  $\kappa$  is merely strictly negative at  $z = 0$ .

**Theorem 3.1**

Let  $\kappa : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}$  be strictly negative at  $z = 0$  and  $u : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}$  a  $C^2$ -solution to  $\Delta u = -\kappa(z) e^{2u}$  in  $\mathbb{D} \setminus \{0\}$ . Then  $\alpha \leq 1$ , where  $\alpha$  is the order of  $u$ , i.e.

$$\alpha := \lim_{r \rightarrow 0} \frac{M_u(r)}{\log(1/r)} \quad (3.1)$$

and

$$u(z) = -\alpha \log |z| + v(z), \quad \text{if } \alpha < 1, \quad (3.2)$$

$$u(z) = -\log |z| - \log \log(1/|z|) + w(z), \quad \text{if } \alpha = 1, \quad (3.3)$$

where  $v(z)$  is continuous at  $z = 0$  and  $w(z) = O(1)$  as  $z \rightarrow 0$ .

Note that the limit  $\alpha$  in (3.1) always exists and  $\alpha > -\infty$  since  $u$  is subharmonic in a punctured neighborhood of  $z = 0$  (see (2.1)).

**Remark 3.2**

Theorem 3.1 was proved before by Heins [12] for  $\kappa(z) \equiv -4$ . It was observed by McOwen [19] that Heins' method can also be used to prove Theorem 3.1 in the general case. We give a different proof below which is easier in many respects. This proof also illustrates in a simple situation the technique we use to establish Theorem 1.1. Moreover, it allows us to say more about the remainder function  $w(z)$  in (3.3) under some mild extra assumptions on the curvature function  $\kappa(z)$ ; cf. Theorem 3.4. These extra information will be crucial in geometric applications such as Theorem 1.2 and Theorem 1.4. An inspection of the method of Heins and McOwen shows that their technique does not seem to be capable of yielding such refinements of Theorem 3.1. We also note that Theorem 3.1 only deals with the case of strictly negative curvature. Positive curvature functions appear to be more difficult to handle and some partial results in this case like one-sided estimates have recently been obtained by Yunyan [34].

**Proof of Theorem 3.1.**

Without loss of generality we may assume  $u \in C(\overline{\mathbb{D}} \setminus \{0\})$  and

$$-a \leq \kappa(z) \leq -A \quad \text{in } \mathbb{D} \setminus \{0\} \quad (3.4)$$

for some finite constants  $a > 0$  and  $A > 0$ . We first show that the order  $\alpha$  of  $u(z)$  at  $z = 0$  is always  $\leq 1$ . To see this just note that  $u$  is a subsolution to  $\Delta v = A e^{2v}$  in  $\mathbb{D} \setminus \{0\}$ , so

$$u(z) \leq \log \left( \frac{1}{\sqrt{A}} \frac{1}{|z| \log(1/|z|)} \right), \quad z \in \mathbb{D} \setminus \{0\}, \quad (3.5)$$

because the function on the right-hand side is the maximal solution to  $\Delta\nu = A e^{2\nu}$  in  $\mathbb{D}\setminus\{0\}$ . This follows from Ahlfors' lemma [1] by noting that

$$\frac{|dz|}{\sqrt{A}|z|\log(1/|z|)}$$

is the hyperbolic metric of the punctured disk  $\mathbb{D}\setminus\{0\}$  with constant curvature  $-A$ . Now, (3.1) is an immediate consequence of the inequality (3.5).

In a next step we construct a supersolution to  $\Delta u = -\kappa(z) e^{2u}$  by looking for a solution  $u_\alpha^A$  to  $\Delta\nu = A e^{2\nu}$  with order  $\alpha$  at  $z = 0$ . Indeed it is not difficult to show that

$$u_\alpha^A(z) := \begin{cases} \log\left(\frac{2}{\sqrt{A}} \frac{(1-\alpha)|z|^{-\alpha}}{1-|z|^{2(1-\alpha)}}\right) & \text{if } \alpha < 1, \\ \log\left(\frac{1}{\sqrt{A}} \frac{1}{|z|\log(1/|z|)}\right) & \text{if } \alpha = 1, \end{cases} \quad (3.6)$$

has the required properties. For  $\alpha < 1$  this supersolution is obtained by noting that the conformal Riemannian metric

$$\lambda_\alpha^A(z) |dz| := e^{u_\alpha^A(z)} |dz|$$

is the formal pullback of the hyperbolic metric of the unit disk with constant curvature  $-A$

$$\frac{2}{\sqrt{A}} \frac{|dz|}{1-|z|^2},$$

under the map  $z \mapsto z^{1-\alpha}$ . We are now in a position to apply the extended maximum principle (Theorem 2.4) which leads to

$$u(z) \leq u_\alpha^A(z) \quad \text{for } z \in \mathbb{D}\setminus\{0\}. \quad (3.7)$$

In order to get a lower bound for  $u(z)$ , we next look for appropriate subsolutions to  $\Delta u = -\kappa(z) e^{2u}$ , i.e., in view of (3.4), for solutions to  $\Delta\nu = a e^{2\nu}$  with order  $\alpha$  at  $z = 0$ . A one-parameter family of such solutions  $u_{\alpha,R}^a$ ,  $R > 1$ , is obtained from the functions  $u_\alpha^A$  in (3.6) by replacing  $A$  with  $a$  and a suitable rescaling:

$$u_{\alpha,R}^a(z) := u_\alpha^a(z/R) + \log(1/R).$$

Geometrically, the conformal Riemannian metric  $e^{u_{\alpha,R}^a(z)} |dz|$  is the pullback of the hyperbolic metric

- (i) of the unit disk  $\mathbb{D}$  with constant curvature  $-a$  under the map  $z \mapsto (z/R)^{1-\alpha}$  if  $\alpha < 1$ ;

and

- (ii) of the punctured unit disk  $\mathbb{D}\setminus\{0\}$  with constant curvature  $-a$  under the map  $z \mapsto z/R$  if  $\alpha = 1$ .

Thus,  $u_{\alpha,R}^a$  is a subsolution to  $\Delta u = -\kappa(z) e^{2u}$  in  $\mathbb{D}\setminus\{0\}$  with order  $\alpha$  at  $z = 0$  for each  $R > 1$ . We now have to choose the parameter  $R > 1$  in an appropriate way. For this denote

$$m := \min_{|z|=1} u(z).$$

Observe that for every  $|z| = 1$

$$\lim_{R \rightarrow \infty} u_{\alpha,R}^a(z) = \lim_{R \rightarrow \infty} u_{\alpha,R}^a(1) = -\infty.$$

Consequently, we can find  $R > 1$  such that  $u_{\alpha,R}^a(z) \leq m$  for  $|z| = 1$ . Hence, we can again apply the extended maximum principle and obtain

$$u_{\alpha,R}^a(z) \leq u(z) \quad \text{for } z \in \mathbb{D} \setminus \{0\}. \quad (3.8)$$

Combining (3.7) and (3.8) yields

$$\log \left( \frac{2(1-\alpha)}{\sqrt{a}} \frac{1}{R^{1-\alpha}(1-|z/R|^{2(1-\alpha)})} \right) \leq u(z) + \alpha \log |z| \leq \log \left( \frac{2(1-\alpha)}{\sqrt{A}} \frac{1}{1-|z|^{2(1-\alpha)}} \right)$$

for  $\alpha < 1$  and

$$\log \left( \frac{1}{\sqrt{a}} \frac{\log(1/|z|)}{\log(R/|z|)} \right) \leq u(z) + \log |z| + \log \log \frac{1}{|z|} \leq \log \left( \frac{1}{\sqrt{A}} \right)$$

for  $\alpha = 1$ . Thus  $u$  has the desired representation (3.2) and (3.3) respectively, where the remainder functions  $v$  and  $w$  are continuous in  $\mathbb{D} \setminus \{0\}$  and bounded at  $z = 0$ .

To complete the proof of Theorem 3.1, we need to show that for  $\alpha < 1$  the remainder function  $v(z)$  is in fact continuous at  $z = 0$ . Using the PDE  $\Delta u = -\kappa(z) e^{2u}$  we get that  $v(z) = u(z) + \alpha \log |z|$  is a solution to

$$\Delta v = \frac{-\kappa(z)}{|z|^{2\alpha}} e^{2v} \quad \text{for } z \in \mathbb{D} \setminus \{0\},$$

which shows that  $v$  is subharmonic on  $\mathbb{D} \setminus \{0\}$ . On the other hand  $v$  is bounded in a neighborhood of  $z = 0$  and consequently subharmonic in all of  $\mathbb{D}$ . The fact that  $z \mapsto \Delta v(z)$  is integrable over  $K_r$  for each  $0 < r < 1$  legitimizes the use of Proposition 4.1 and leads to

$$v(z) = h(z) + \frac{1}{2\pi} \iint_{K_r} \log |z - \xi| \frac{-\kappa(\xi)}{|\xi|^{2\alpha}} e^{2v(\xi)} d\sigma_\xi, \quad z \in K_r, \quad (3.9)$$

where  $h$  is a harmonic function on  $K_r$ . The continuity of  $v$  in all of  $\mathbb{D}$  is now an immediate consequence of Proposition 4.2. ■

In Theorem 3.1 only the fact that  $\kappa$  is strictly negative at  $z = 0$  guarantees the continuity of the remainder function  $v$  at  $z = 0$  for  $\alpha < 1$ . This is no longer true if  $\alpha = 1$  as the following example shows.

### Example 3.3

*The function*

$$u(z) = -\log |z| - \log \log \frac{1}{|z|} + 2 + \frac{\sin(\log \log(1/|z|))}{6 + \sin(\log \log(1/|z|))}$$

*is a solution to*

$$\Delta u = -\kappa(z) e^{2u}$$

*in  $\mathbb{D} \setminus \{0\}$ , where*

$$\kappa(z) = \left( 6 \frac{\sin \beta(z) + \cos \beta(z)}{(6 + \sin \beta(z))^2} + 12 \frac{(\cos \beta(z))^2}{(6 + \sin \beta(z))^3} - 1 \right) \exp \left( -6 + \frac{12}{6 + \sin(\beta(z))} \right),$$

*with  $\beta(z) = \log \log(1/|z|)$ . Obviously, both the curvature  $\kappa(z)$  and the remainder function  $w(z)$  are bounded when  $z \rightarrow 0$  but not continuous at  $z = 0$ .*

Thus the case  $\alpha = 1$  is exceptional as far as the continuity of the remainder function  $w$  is concerned. We now show that the remainder function  $w$  is continuous at  $z = 0$ , when we additionally assume that the curvature  $\kappa$  is continuous at  $z = 0$ .

**Theorem 3.4**

Let  $\kappa : \mathbb{D} \rightarrow \mathbb{R}$  be a continuous function with  $\kappa(0) < 0$ . If  $u(z) = -\log |z| - \log \log(1/|z|) + w(z)$ , where  $w(z) = O(1)$  for  $z \rightarrow 0$ , is a solution to  $\Delta u = -\kappa(z) e^{2u}$  in  $\mathbb{D} \setminus \{0\}$ , then  $w$  is continuous at  $z = 0$  and  $w(0) = -\log \sqrt{-\kappa(0)}$ .

We wish to point out that in contrast to the cases  $\alpha < 1$ , where the continuity of the remainder function  $v$  follows from potential-theoretical considerations under weaker assumptions, this method does not work in the exceptional case  $\alpha = 1$ , even though there is a corresponding representation formula for the remainder function  $w$  (see formula (3.13) below). Instead, we use the method of super- and subsolutions and the generalized maximum principle (Theorem 2.4) for this purpose.

**Proof of Theorem 3.4.**

Set  $b := -\kappa(0)$  and choose  $0 < \varepsilon < b/2$ . By the continuity of  $\kappa$  there is a disk  $K_\varrho$  such that

$$0 < b - \varepsilon \leq -\kappa(z) \leq b + \varepsilon \quad \text{for } z \in K_\varrho.$$

Next, define

$$R = \varrho \exp \left( 1 / \left( \varrho \sqrt{b} \tilde{m}_\varrho \right) \right) \quad \text{and} \quad R' = \varrho \exp \left( 1 / \left( \varrho \sqrt{b} \tilde{M}_\varrho \right) \right),$$

where

$$\tilde{M}_\varrho = \max_{|z|=\varrho} e^{u(z)} \quad \text{and} \quad \tilde{m}_\varrho = \min_{|z|=\varrho} e^{u(z)},$$

and let

$$u_-(z) := \log \left( \frac{1}{\sqrt{b+\varepsilon}} \frac{1}{|z| \log(R/|z|)} \right) \quad \text{and} \quad u_+(z) := \log \left( \frac{1}{\sqrt{b-\varepsilon}} \frac{1}{|z| \log(R'/|z|)} \right).$$

Then  $u_-$  is a subsolution to  $\Delta u = -\kappa(z) e^{2u}$  in  $K_\varrho \setminus \{0\}$  with order 1 at  $z = 0$  and  $u_- \leq u$  on  $|z| = \varrho$ . Also,  $u_+$  is a supersolution to  $\Delta u = -\kappa(z) e^{2u}$  in  $K_\varrho \setminus \{0\}$  with order 1 at  $z = 0$  and  $u \leq u_+$  on  $|z| = \varrho$ . By the extended maximum principle (Theorem 2.4), we get

$$\log \left( \frac{1}{\sqrt{b+\varepsilon}} \frac{1}{|z| \log(R/|z|)} \right) \leq u(z) \leq \log \left( \frac{1}{\sqrt{b-\varepsilon}} \frac{1}{|z| \log(R'/|z|)} \right) \quad \text{for } z \in K_\varrho \setminus \{0\}.$$

Rearranging the latter inequality yields

$$w(z) \rightarrow \log \left( \frac{1}{\sqrt{b}} \right) \quad \text{for } z \rightarrow 0,$$

which is what had to be proved. ■

### 3.2 First derivatives of the remainder functions: Results

We now turn to a discussion of the properties of the first partial derivatives of the remainder functions  $v$  and  $w$ . We first consider the case of strictly negative curvature functions.

**Theorem 3.5**

Let  $\kappa : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}$  be strictly negative at  $z = 0$  and let  $u : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}$  be a  $C^2$ -solution to  $\Delta u = -\kappa(z) e^{2u}$  in  $\mathbb{D} \setminus \{0\}$  with order  $\alpha \leq 1$  at  $z = 0$ , i.e.,  $u(z) := -\alpha \log |z| + v(z)$  for  $\alpha < 1$  and  $u(z) := -\log |z| - \log \log(1/|z|) + w(z)$  for  $\alpha = 1$ . Then

$$\begin{aligned} & \text{and} \quad v_z(z), v_{\bar{z}}(z) \text{ are continuous at } z = 0 && \text{if } \alpha < 1/2; \\ & v_z(z), v_{\bar{z}}(z) = O(\log(1/|z|)) && \text{if } \alpha = 1/2, \\ & v_z(z), v_{\bar{z}}(z) = O(|z|^{1-2\alpha}) && \text{if } 1/2 < \alpha < 1, \\ & w_z(z), w_{\bar{z}}(z) = O(|z|^{-1}(\log(1/|z|))^{-1}) && \text{if } \alpha = 1, \end{aligned}$$

when  $z$  approaches  $z = 0$ .

**Remark 3.6**

- (a) Theorem 3.5 is sharp. In fact, for  $\alpha \notin \{1/2, 1\}$ , this is illustrated with Example 2.1. For  $\alpha = 1/2$  see Example 3.9, and for  $\alpha = 1$  one can use the function of Example 3.3.
- (b) We also note that if  $\alpha \neq 1/2$  and  $\alpha \neq 1$ , then Theorem 3.5 refines (part of) Theorem 1.1.
- (c) In view of (a) and (b) we see that  $\alpha = 1/2$  and  $\alpha = 1$  are exceptional cases as far as the first derivatives of the remainder functions  $v$  and  $w$  are concerned. Also, in order to prove the (stronger) assertions of Theorem 1.1 for the first derivatives of  $v$  and  $w$  in those exceptional cases, it does not suffice to assume that  $\kappa$  is strictly negative at  $z = 0$ . As we shall see in Example 3.10 (for  $\alpha = 1/2$ ) and Example 3.11 (for  $\alpha = 1$ ) it is even not enough to suppose that  $\kappa$  is continuous (and negative) at  $z = 0$ .

In the following theorem we therefore adopt the standard assumption for second order elliptic PDEs and consider curvature functions  $\kappa$  which are Hölder continuous on  $\mathbb{D}$ . This assumption was used before for the curvature equation for instance by Troyanov [30, p. 800], but see also the remarks following the next theorem.

**Theorem 3.7**

If  $\kappa$  is locally Hölder continuous in  $\mathbb{D}$  with  $\kappa(0) < 0$ , then with the notation of Theorem 3.5,

$$\begin{aligned} & \text{and} \quad v_z(z), v_{\bar{z}}(z) = O(1) && \text{if } \alpha = 1/2, \\ & w_z(z), w_{\bar{z}}(z) = O(|z|^{-1}(\log(1/|z|))^{-2}) && \text{if } \alpha = 1, \end{aligned}$$

when  $z$  tends to  $z = 0$ .

By Example 2.1, Theorem 3.7 is best possible. We note that our proof of the case  $\alpha = 1/2$  in Theorem 3.7 only uses potential-theoretic tools. For the case  $\alpha = 1$ , by contrast, potential theory alone is not enough. We require further *a-priori* information about the solutions to the curvature equation in this case and we are going to use our extended maximum principle for this purpose. However, in order to be able to apply the extended maximum principle we need to construct suitable sub- and supersolutions to  $\Delta u = -\kappa(z) e^{2u}$ , which take the Hölder continuity of the curvature function  $\kappa$  into account. This is the key point. It turns out that it is easier to construct the required sub- and supersolution under weaker assumptions on the curvature function.<sup>3</sup> We have the following crucial lemma.

---

<sup>3</sup>Die Vergrößerung der Beweislast kann also vorteilhaft sein: denn sie stärkt den Beweisträger [25].

**Lemma 3.8**

Let  $\kappa : \mathbb{D} \rightarrow \mathbb{R}$  be a function with  $\kappa(0) < 0$  and

$$\kappa(z) = \kappa(0) + \frac{r(z)}{(\log(1/|z|))^2}, \quad (3.10)$$

where  $r(z) = O(1)$  as  $z \rightarrow 0$ . If  $u : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}$  is a solution to  $\Delta u = -\kappa(z) e^{2u}$  in  $\mathbb{D} \setminus \{0\}$  with  $u(z) = -\log|z| - \log \log(1/|z|) + w(z)$  where  $w(z) = O(1)$  for  $z \rightarrow 0$ , then there is a disk  $K_\varrho$  such that

$$\left| -\kappa(z) e^{2w(z)} - 1 \right| \leq \frac{C}{\log(1/|z|)}, \quad z \in K_\varrho, \quad (3.11)$$

for some constant  $C > 0$ .

Clearly, every Hölder continuous curvature function  $\kappa : \mathbb{D} \rightarrow \mathbb{R}$  fulfills condition (3.10) of Lemma 3.8. The estimate (3.11) is exactly the *a-priori* information we need to know about the remainder function  $w$  in order to start up the potential-theoretic part of the proof of Theorem 3.7 for  $\alpha = 1$ . We also note that Lemma 3.8 is in a certain sense a higher-order version of the classical Ahlfors' lemma [1].

**3.3 First derivatives of the remainder functions: Proofs and examples**

We now give the proofs of the results of Paragraph 3.2 and illustrate their sharpness with a number of examples. It suffices to restrict ourselves to the investigation of the derivative with respect to  $z$ .

We start with the proof of Theorem 3.5 for  $\alpha < 1$ .

**Proof of Theorem 3.5 for  $\alpha < 1$ .**

Without loss of generality we may assume  $-a \leq \kappa(z) \leq -A$  in  $\mathbb{D}$ , where  $a$  and  $A$  are positive constants. We now make essential use of the representation formula (3.9) for the remainder function  $v(z)$ , which has been established in the course of the proof of Theorem 3.1.

If  $\alpha < 1/2$  the claim is an immediate consequence of (3.9) and Proposition 4.2.

For  $1/2 \leq \alpha < 1$ , pick  $R < 1$ . Then formula (3.9) combined with Proposition 4.2 gives

$$|v_z(z)| \leq \sup_{z \in K_R} |h_z(z)| + \sup_{z \in K_R} \left( -\kappa(z) e^{2v(z)} \right) \frac{1}{2\pi} \iint_{K_R} \frac{1}{2} \frac{1}{|z - \xi|} \frac{1}{|\xi|^{2\alpha}} d\sigma_\xi, \quad z \in K_R \setminus \{0\}.$$

where  $h$  is a harmonic function on  $K_R$ . To see that

$$v_z(z) = \begin{cases} O(\log(1/|z|)) & \text{if } \alpha = 1/2, \\ O(|z|^{1-2\alpha}) & \text{if } 1/2 < \alpha < 1, \end{cases}$$

we just note that

$$\frac{1}{2\pi} \iint_{K_R} \frac{1}{|z - \xi|} \frac{1}{|\xi|} d\sigma_\xi \leq C_1 + \log \frac{1}{|z|}, \quad \text{if } \alpha = \frac{1}{2},$$

and

$$\frac{1}{2\pi} \iint_{K_R} \frac{1}{|z - \xi|} \frac{1}{|\xi|^{2\alpha}} d\sigma_\xi \leq C_2 |z|^{1-2\alpha}, \quad \text{if } \frac{1}{2} < \alpha < 1,$$

(see [9, p. 215]) for some positive constants  $C_1$  and  $C_2$ . ■

If the curvature  $\kappa$  is strictly negative at  $z = 0$ , but not necessarily continuous, then in the preceding proof the estimate for the case  $\alpha = 1/2$  is best possible:

**Example 3.9**

Let  $\kappa : \mathbb{D} \setminus \{0\} \rightarrow [-6e, -4e^{-8}]$  be given by

$$z \mapsto - \left( 4 + 2 \frac{|\operatorname{Re}(z)|}{|z|} \right) \exp(-2(|\operatorname{Re}(z)| \log |z| + 4|z|)) .$$

Then the function  $u(z) = -1/2 \log |z| + v(z)$  where  $v(z) = |\operatorname{Re}(z)| \log |z| + 4|z|$  is a solution to

$$\Delta u = -\kappa(z) e^{2u} \quad \text{in } \mathbb{D} \setminus \{0\}$$

and

$$v_z(z) = \frac{|\operatorname{Re}(z)|}{2z} + 2 \frac{\bar{z}}{|z|} - \frac{\operatorname{Re}(z)}{2|\operatorname{Re}(z)|} \log(1/|z|) .$$

**Proof of Theorem 3.7 for  $\alpha = 1/2$ .**

At first we observe the assumption that  $\kappa$  is locally Hölder continuous in  $\mathbb{D}$  (without loss of generality with fixed exponent  $\gamma$ ,  $0 < \gamma \leq 1$ , say) implies that  $v$  is locally Hölder continuous in  $\mathbb{D}$ . To this end we may suppose  $-a \leq \kappa(z) \leq -A$  in  $\mathbb{D}$ , where  $a$  and  $A$  are positive constants. Further, we define in  $\mathbb{D} \setminus \{0\}$  the regular conformal Riemannian metric  $\mu(z) |dz| = e^{u(z)} |dz|$  and write  $\mu(z) = |z|^{-1/2} \sigma(z)$ , so  $\sigma(z) = e^{v(z)}$  is continuous at  $z = 0$  and  $\sigma(0) > 0$ ; see Theorem 3.1. Now

$$\tilde{\mu}(z) |dz| := 2\mu(z^2) |z| |dz| = 2\sigma(z^2) |dz|$$

is a (positive) continuous conformal Riemannian metric in  $\mathbb{D}$  which is regular  $\mathbb{D} \setminus \{0\}$ . Its curvature  $\kappa_{\tilde{\mu}}$  is locally Hölder continuous in  $\mathbb{D}$  as  $\kappa_{\tilde{\mu}}(z) = \kappa(z^2)$ . Thus  $\tilde{\mu}(z)$  has a  $C^2$ -extension to  $\mathbb{D}$  by elliptic regularity. In particular,  $\tilde{\mu}_z$  is bounded in  $K_R$  for some  $R < 1$ . By construction, this implies

$$|z \cdot v_z(z)| \leq C |z|^{1/2}$$

for  $z \in K_R$  and some constant  $C$ , so  $v$  is locally Hölder continuous in  $\mathbb{D}$  with exponent  $1/2$ .

This in turn implies that the function  $-\kappa(\xi) e^{2v(\xi)}$  is locally Hölder continuous in  $\mathbb{D}$  with exponent  $\tilde{\gamma} = \min\{\gamma, 1/2\}$  and

$$\kappa(\xi) e^{2v(\xi)} = \kappa(0) e^{2v(0)} + r(\xi) |\xi|^{\tilde{\gamma}} \quad \text{for } \xi \in \mathbb{D},$$

where  $r$  is a continuous function on  $\mathbb{D} \setminus \{0\}$  with  $r(\xi) = O(1)$  for  $\xi \rightarrow 0$ . Thus we have for  $z \in K_R \setminus \{0\}$  in view of formula (3.9) and Proposition 4.2

$$v_z(z) = h_z(z) + \frac{1}{2\pi} \iint_{K_R} \frac{1}{2(z-\xi)} \frac{-\kappa(0) e^{2v(0)}}{|\xi|} d\sigma_\xi - \frac{1}{2\pi} \iint_{K_R} \frac{1}{2(z-\xi)} \frac{r(\xi)}{|\xi|^{1-\tilde{\gamma}}} d\sigma_\xi, \quad (3.12)$$

where  $h$  is harmonic on  $K_R$ .

To confirm that  $v_z(z) = O(1)$  as  $z \rightarrow 0$ , we first observe that the second integral of (3.12) is bounded at  $z = 0$ . For the first integral of (3.12) we define  $k := -\kappa(0) e^{2v(0)}$  and consider



the subharmonic function  $\omega(z) = k|z|$  on  $\mathbb{D}$ . Since  $\Delta\omega(z) = k/|z|$  for  $z \in \mathbb{D} \setminus \{0\}$  we have by Proposition 4.1

$$\omega(z) = h_\omega(z) + \frac{1}{2\pi} \iint_{K_R} \log |z - \xi| \frac{k}{|\xi|} d\sigma_\xi, \quad z \in K_R,$$

where  $h_\omega$  is harmonic on  $K_R$ . Lastly, differentiating yields

$$\omega_z(z) = k \frac{\bar{z}}{|z|} \quad \text{and} \quad \omega_z(z) = h_{\omega_z}(z) + \frac{1}{2\pi} \iint_{K_R} \frac{1}{2(z - \xi)} \frac{k}{|\xi|} d\sigma_\xi$$

for  $z \in K_R \setminus \{0\}$ . A comparison with (3.12) completes the proof.  $\blacksquare$

For  $\alpha = 1/2$  we note that if  $\kappa$  is merely continuous at  $z = 0$ , then  $v_z$  is not necessarily bounded at  $z = 0$ :

**Example 3.10**

Consider the function  $u : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}$  defined by

$$u(z) = -\frac{1}{2} \log |z| + \operatorname{Re}(z) \log \log(e/|z|) + 4|z|.$$

Then  $u$  is a solution to

$$\Delta u = -\kappa(z) e^{2u}$$

in  $\mathbb{D} \setminus \{0\}$ , where

$$\kappa(z) = - \left( 4 + \frac{\operatorname{Re}(z)}{|z|} \frac{-3 + 2 \log |z|}{(1 - \log |z|)^2} \right) \exp(-2(\operatorname{Re}(z) \log \log(e/|z|) + 4|z|)).$$

Note that  $\kappa(z)$  is continuous in  $\mathbb{D}$  with  $\kappa(0) = -4$  but not locally Hölder continuous at  $z = 0$ . For the derivative of

$$v(z) = \operatorname{Re}(z) \log \log(e/|z|) + 4|z|$$

we obtain

$$v_z(z) = 2 \frac{\bar{z}}{|z|} - \frac{1}{2} \frac{\operatorname{Re}(z) \bar{z}}{|z|^2 \log(e/|z|)} + \frac{1}{2} \log \log(e/|z|).$$

Thus  $|v_z(z)| \rightarrow \infty$  for  $z \rightarrow 0$ .

**Proof of Theorem 3.5 for  $\alpha = 1$ .**

(i) We first show that for every  $0 < r < 1$

$$w(z) = h(z) + \frac{1}{2\pi} \iint_{K_r} \log |z - \xi| \frac{-\kappa(\xi) e^{2w(\xi)} - 1}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi \quad \text{for } z \in K_r \setminus \{0\}, \quad (3.13)$$

where  $h$  is a harmonic function on  $K_r$ . This is the analogue to the corresponding representation formula (3.9) for the function  $v$  for the cases  $\alpha < 1$ .

In order to prove (3.13) we define for  $z \in K_r \setminus \{0\}$  the function

$$p(z) := w(z) - \log \log(1/|z|).$$

Since

$$\Delta p(z) = \frac{-\kappa(z)}{|z|^2 (\log(1/|z|))^2} e^{2w(z)} > 0 \quad \text{for } z \in K_r \setminus \{0\}$$

and  $\lim_{z \rightarrow 0} p(z) = -\infty$ ,  $p$  is subharmonic on  $K_r$ .

By Proposition 4.1, as  $z \mapsto \Delta p(z)$  is integrable over  $K_r$ , we obtain

$$p(z) = h_p(z) + \frac{1}{2\pi} \iint_{K_r} \log |z - \xi| \frac{-\kappa(\xi) e^{2w(\xi)}}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi, \quad z \in K_r,$$

where  $h_p$  is harmonic on  $K_r$ .

Applying Proposition 4.1 again, this time to the subharmonic function  $q(z) := -\log \log(1/|z|)$ , we deduce that

$$q(z) = h_q(z) + \frac{1}{2\pi} \iint_{K_r} \log |z - \xi| \frac{1}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi \quad \text{for } z \in K_r,$$

where  $h_q$  is harmonic on  $K_r$ . This gives (3.13) with the harmonic function  $h(z) = h_p(z) - h_q(z)$ .

(ii) Let  $R < 1/(2e^4)$  and set  $q(z) = -\kappa(z) e^{2w(z)} - 1$ . According to the representation formula (3.13) and Proposition 4.2 we obtain

$$|w_z(z)| \leq \sup_{z \in K_R} |h_z(z)| + \frac{1}{2\pi} \iint_{K_R} \frac{1}{2} \frac{1}{|z - \xi|} \frac{|q(\xi)|}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi, \quad z \in K_R \setminus \{0\},$$

where  $h$  is a harmonic function on  $K_R$ . We only have to find that for small  $z$ ,

$$I(z) := \frac{1}{2\pi} \iint_{K_R} \frac{1}{|z - \xi|} \frac{|q(\xi)|}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi \leq \frac{C}{|z| \log(1/|z|)}$$

for an appropriate constant  $C$ . To this end fix  $z \in K_{R/2}$  and set  $r = |z|/2$ . Further, let  $N = \{\xi \in K_R \setminus (K_r \cup K_r(z)) : |z - \xi| \geq |\xi|\}$  and  $P = \{\xi \in K_R \setminus (K_r \cup K_r(z)) : |z - \xi| < |\xi|\}$ , where  $K_r(z)$  stands for the disk about  $z$  with radius  $r$ . Then, if  $M = \sup_{\xi \in K_R} |q(\xi)|$ ,

$$\begin{aligned} I(z) &\leq \frac{1}{2\pi} \iint_{K_r} \frac{1}{|z - \xi|} \frac{|q(\xi)|}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi + \frac{1}{2\pi} \iint_{K_r(z)} \frac{1}{|z - \xi|} \frac{M}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi \\ &\quad + \frac{1}{2\pi} \iint_N \frac{1}{|z - \xi|} \frac{M}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi + \frac{1}{2\pi} \iint_P \frac{1}{|z - \xi|} \frac{M}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi \\ &\leq \sup_{\xi \in K_r} |q(\xi)| \frac{1}{r} \int_0^r \frac{1}{\varrho (\log(1/\varrho))^2} d\varrho + \frac{M}{r^2 (\log(1/r))^2} \int_0^r d\varrho \\ &\quad + \frac{M}{\sqrt{r} (\log(1/r))^2} \int_r^R \frac{1}{\varrho^{3/2}} d\varrho + \frac{M}{\sqrt{r} (\log(1/r))^2} \int_r^{2R} \frac{1}{\varrho^{3/2}} d\varrho \\ &\leq \sup_{\xi \in K_r} |q(\xi)| \frac{2}{|z| \log(1/|z|)} + \frac{10M}{|z| (\log(1/|z|))^2} = \frac{C}{|z| \log(1/|z|)} \end{aligned} \tag{3.14}$$

for some constant  $C > 0$ . ■

For strictly negative  $\kappa$  (not necessarily continuous) the above proof gives an optimal estimate as it can be seen from Example 3.3. If  $\kappa$  is continuous in  $\mathbb{D}$ , then the function  $q(z) = -\kappa(z) e^{2w(z)} - 1$  is continuous at  $z = 0$  with  $q(0) = 0$  by Theorem 3.4. Thus  $w_z(z) = o(|z|^{-1} \log(1/|z|)^{-1})$  by (3.14). This is pretty sharp, because the following example shows that  $|w_z(z)| = C \cdot (|z|^{-1} (\log(1/|z|))^{-1-\beta})$  is possible for any  $\beta > 0$ .

**Example 3.11**

Define for  $\beta > 0$  and  $z \in \mathbb{D}$  the continuous function

$$\kappa(z) = -\exp\left(\frac{-2}{(\log(1/|z|))^\beta}\right) \left(1 + \beta(1 + \beta) \frac{1}{(\log(1/|z|))^\beta}\right).$$

Then

$$u(z) = -\log |z| - \log \log \frac{1}{|z|} + \frac{1}{(\log(1/|z|))^\beta}, \quad z \in \mathbb{D} \setminus \{0\},$$

is a solution to

$$\Delta u = -\kappa(z) e^{2u}$$

in  $\mathbb{D} \setminus \{0\}$ . Further, for the remainder function  $w(z) = (\log(1/|z|))^{-\beta}$  we see that

$$w_z(z) = \frac{\beta}{2} \frac{1}{z} \frac{1}{(\log(1/|z|))^{\beta+1}}.$$

On the other hand, if  $\kappa$  is Hölder continuous in  $\mathbb{D}$ , then the potential-theoretic estimate (3.14) combined with Lemma 3.8 immediately proves Theorem 3.7 for the case  $\alpha = 1$ . So to finish the proof of Theorem 3.7 we finally need to prove Lemma 3.8.

**Proof of Lemma 3.8.**

Without loss of generality we may suppose  $\kappa(0) = -1$ . Further, it is advantageous to take the regular conformal Riemannian metric  $\lambda(z) |dz| = e^{u(z)} |dz| = (|z| \log(1/|z|))^{-1} \sigma(z) |dz|$ , where  $0 < \liminf_{z \rightarrow 0} \sigma(z) \leq \limsup_{z \rightarrow 0} \sigma(z) < \infty$  with curvature  $\kappa$  into account. In a first step we establish

$$C_1 \left(\log \frac{1}{|z|}\right)^{-1} \leq \lambda(z) |z| \log \frac{1}{|z|} - 1 \leq C_2 \left(\log \frac{1}{|z|}\right)^{-1} \quad (3.15)$$

for all  $z$  in some disk  $K_\varrho$  and constants  $C_1 < 0 < C_2$  by comparing  $\lambda(z) |dz|$  with suitable conformal Riemannian metrics.

For the moment we define for arbitrary  $R > 0$  on  $K_R \setminus \{0\}$  the conformal Riemannian metrics

$$\lambda_R^a(z) |dz| = \frac{\exp(a(\log(R/|z|))^{-1})}{|z| \log(R/|z|)} |dz|, \quad a > 0,$$

and

$$\lambda_R(z) |dz| = \frac{\exp((\log(1/|z|))^{-1})}{|z| \log(R/|z|)} |dz|$$

with curvature

$$\kappa_{\lambda_R^a}(z) = -1 + 2a^2(\log(1/|z|))^{-2} + O((\log(1/|z|))^{-3})$$

and

$$\kappa_{\lambda_R}(z) = -1 + 2(1 - 2 \log R) (\log(1/|z|))^{-2} + O((\log(1/|z|))^{-3})$$

respectively. We observe that

- (i)  $\lambda_R^a(z) \rightarrow +\infty$  as  $R \searrow |z|$  for fixed  $|z|$ ;
- (ii)  $\lambda_R(z) \rightarrow 0$  as  $R \rightarrow \infty$  for fixed  $z$ ;
- (iii)  $R \mapsto \kappa_{\lambda_R^a}(z)$  and  $R \mapsto \kappa_{\lambda_R}(z)$  are monotonically decreasing.

Now, choosing  $a$  and  $R$  appropriately will lead to (3.15).

For that we first find  $C > 0$  and  $\varrho > 0$  such that

$$-C \leq r(z) \leq C, \quad z \in K_\varrho.$$

To derive the right inequality of (3.15), fix  $a > 0$  with  $a^2 \geq C$ . Then, shrinking  $\varrho$  if necessary, we deduce that

$$-\kappa(z) = 1 - r(z) \left( \log \frac{1}{|z|} \right)^{-2} \geq -\kappa_{\lambda_1^a}(z) + o((\log(1/|z|))^{-2}) + a^2 \left( \log \frac{1}{|z|} \right)^{-2} \geq -\kappa_{\lambda_1^a}(z)$$

for  $z \in K_\varrho$ . The monotonicity of  $\kappa_{\lambda_R^a}$  implies now that for all  $R$  with  $\varrho < R < 1$

$$-\kappa(z) \geq -\kappa_{\lambda_R^a}(z), \quad z \in K_\varrho.$$

So  $z \mapsto \log \lambda_R^a(z)$  is a supersolution to  $\Delta u = -\kappa(z) e^{2u}$  in  $K_\varrho \setminus \{0\}$  for every  $R \in (\varrho, 1)$ . Further, by (i) we choose  $R \in (\varrho, 1)$  such that  $\lambda(z) \leq \lambda_R^a(z)$  for  $|z| = \varrho$ . Therefore by Theorem 2.4 we get

$$\lambda(z) \leq \lambda_R^a(z) \quad \text{for } z \in K_\varrho \setminus \{0\} \quad (3.16)$$

and finally

$$\lambda(z) |z| \log(1/|z|) \leq \frac{\log(1/|z|)}{\log(R/|z|)} \exp(a(\log(R/|z|))^{-1}) = 1 + (a - \log R) \frac{1}{\log(1/|z|)} + \dots$$

The proof for the left inequality of (3.15) runs similarly. Here we choose  $R' > 1$  such that  $-(1 - 2 \log R') \geq C$ . Then, again shrinking  $\varrho$  if necessary, we have

$$-\kappa(z) \leq -\kappa_{\lambda_{R'}}(z) + o((\log(1/|z|))^{-2}) + (1 - 2 \log R') \left( \log \frac{1}{|z|} \right)^{-2} \leq -\kappa_{\lambda_{R'}}(z)$$

for  $z \in K_\varrho$ , which shows that for all  $R > R'$

$$-\kappa(z) \leq -\kappa_{\lambda_R}(z), \quad z \in K_\varrho.$$

Thus  $\log \lambda_R(z)$  is a subsolution to  $\Delta u = -\kappa(z) e^{2u}$  for every  $R > R'$  and since  $\lambda_R(z) \rightarrow 0$  for  $R \rightarrow \infty$  we can find an  $R > R'$  such that  $\lambda(z) \geq \lambda_R(z)$  for  $|z| = \varrho$ . Using again Theorem 2.4 gives

$$\lambda(z) \geq \lambda_R(z) \quad \text{for } z \in K_\varrho \setminus \{0\} \quad (3.17)$$

and consequently

$$\lambda(z) |z| \log \frac{1}{|z|} \geq \frac{\log(1/|z|)}{\log(R/|z|)} \exp((\log(1/|z|))^{-1}) = 1 + (1 - \log R) \frac{1}{\log(1/|z|)} + \dots$$

This yields the left-hand side of inequality (3.15) for  $z \in K_\varrho$ .

To see that (3.11) holds, just note that  $e^{w(z)} = \lambda(z) |z| \log(1/|z|)$  and

$$-\kappa(z) e^{2w(z)} - 1 = (e^{w(z)} - 1)(e^{w(z)} + 1) - \frac{r(z) e^{2w(z)}}{(\log(1/|z|))^2}.$$

Since  $w(z)$  is bounded on every disk  $K_R$ ,  $R < 1$ , the result follows from (3.15), shrinking  $\varrho$  again, if necessary. ■

### 3.4 Second derivative

We are finally left to establish the statements about the second derivatives of Theorem 1.1. The proof is similar to the case of the first derivatives above. Therefore, we restrict ourselves to indicate the necessary changes in the argument for these derivatives.

#### Proof of Theorem 1.1: Second derivatives.

By assumption  $\kappa$  is locally Hölder continuous in  $\mathbb{D}$  and we may assume a fixed Hölder exponent  $\gamma$ ,  $0 < \gamma \leq 1$ , say. Without loss of generality we may also assume  $-a < \inf_{z \in \mathbb{D}} \kappa(z) < \sup_{z \in \mathbb{D}} \kappa(z) < -A$ , where  $a$  and  $A$  are positive constants and  $\kappa(0) = -1$ .

If  $\alpha \leq 0$  the result follows directly from the representation formula (3.9) and Proposition 4.2.

For  $0 < \alpha < 1$  we define  $q(\xi) = -\kappa(\xi)e^{2v(\xi)}$  and note that  $q$  is locally Hölder continuous in  $\mathbb{D}$  with exponent  $\tilde{\gamma} = \gamma$  if  $0 < \alpha < 1/2$  and  $\tilde{\gamma} = \min\{\gamma, 2 - 2\alpha\}$  if  $1/2 \leq \alpha < 1$ . Next, fix  $R < 1$ , choose  $z \in K_{R/2} \setminus \{0\}$  and set  $r = |z|/2$ . Rearranging (4.1) yields

$$\begin{aligned} \frac{\partial^2}{\partial x_l \partial x_j} v(z) &= \frac{\partial^2}{\partial x_l \partial x_j} h(z) - \frac{1}{2\pi} \frac{q(z)}{|z|^{2\alpha}} \int_{\partial K_r(z)} \frac{\partial}{\partial x_j} \log |z - \xi| n_l(\xi) |d\xi| + \\ &\quad \frac{1}{2\pi} \iint_{K_R \setminus K_r(z)} \frac{\partial^2}{\partial x_l \partial x_j} \log |z - \xi| \frac{q(\xi)}{|\xi|^{2\alpha}} d\sigma_\xi + \frac{1}{2\pi} \iint_{K_r(z)} \frac{\partial^2}{\partial x_l \partial x_j} \log |z - \xi| \left( \frac{q(\xi)}{|\xi|^{2\alpha}} - \frac{q(z)}{|z|^{2\alpha}} \right) d\sigma_\xi \end{aligned}$$

for  $l, j \in \{1, 2\}$ , where  $h$  is harmonic on  $K_R$ . Our aim is to show that the second derivatives of  $v$  belong to  $O(|z|^{-2\alpha})$ . For that we need only observe that

$$\left| \frac{1}{2\pi} \iint_{K_R \setminus K_r(z)} \frac{\partial^2}{\partial x_l \partial x_j} \log |z - \xi| \frac{q(\xi)}{|\xi|^{2\alpha}} d\sigma_\xi \right| \leq \sup_{\xi \in K_R} |q(\xi)| \frac{1}{2\pi} \iint_{K_R \setminus K_r(z)} \frac{2}{|z - \xi|^2} \frac{1}{|\xi|^{2\alpha}} d\sigma_\xi \leq \frac{C_1}{|z|^{2\alpha}}$$

for some constant  $C_1 > 0$  (see [9, p. 215]) and

$$\begin{aligned} &\left| \frac{1}{2\pi} \iint_{K_r(z)} \frac{\partial^2}{\partial x_l \partial x_j} \log |z - \xi| \left( \frac{q(\xi)}{|\xi|^{2\alpha}} - \frac{q(z)}{|z|^{2\alpha}} \right) d\sigma_\xi \right| \leq \\ &\frac{1}{2\pi} \iint_{K_r(z)} \frac{2}{|z - \xi|^2} \frac{|q(\xi) - q(z)|}{|\xi|^{2\alpha}} d\sigma_\xi + \frac{2M}{2\pi} \iint_{K_r(z)} \frac{1}{|z - \xi|^2} \frac{(|\xi|^\alpha + |z|^\alpha) \left| |\xi|^\alpha - |z|^\alpha \right|}{|z|^{2\alpha} |\xi|^{2\alpha}} d\sigma_\xi \leq \frac{C_2}{|z|^{2\alpha}}, \end{aligned}$$

where  $C_2$  is some positive constant.

The case  $\alpha = 1$  runs similarly. At first pick  $R < 1/e^2$ , define  $q(\xi) = -\kappa(\xi)e^{2w(\xi)} - 1$  and put  $M = \sup_{\xi \in K_R} |q(\xi)|$ . Since  $\kappa$  fulfills the hypotheses of Lemma 3.8 condition (3.11) holds for some disk  $K_{\varrho}$ . Now, let  $\tilde{\varrho} = \min\{R/2, \varrho\}$ . Choose  $z \in K_{\tilde{\varrho}}$  and set  $r = |z|/2$ . Rewriting (4.2)

gives

$$\begin{aligned} \frac{\partial^2}{\partial x_l \partial x_j} w(z) &= \frac{\partial^2}{\partial x_l \partial x_j} h(z) + \frac{1}{2\pi} \iint_{K_{\bar{\varrho}} \setminus K_r(z)} \frac{\partial^2}{\partial x_l \partial x_j} \log |z - \xi| \frac{q(\xi)}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi + \\ &\quad \frac{1}{2\pi} \iint_{K_r(z)} \frac{\partial^2}{\partial x_l \partial x_j} \log |z - \xi| \left( \frac{q(\xi)}{|\xi|^2 (\log(1/|\xi|))^2} - \frac{q(z)}{|z|^2 (\log(1/|z|))^2} \right) d\sigma_\xi - \\ &\quad \frac{1}{2\pi} \frac{q(z)}{|z|^2 (\log(1/|z|))^2} \int_{\partial K_r(z)} \frac{\partial}{\partial x_j} \log |z - \xi| n_l(\xi) |d\xi| \end{aligned}$$

for  $l, j \in \{1, 2\}$  and a harmonic function  $h$  on  $K_{\bar{\varrho}}$ . To derive that the second derivatives of  $w$  are in  $O(|z|^{-2}(\log(1/|z|))^{-2})$  as  $|z| \rightarrow 0$ , we first note that

$$\left| \frac{1}{2\pi} \iint_{K_{\bar{\varrho}} \setminus K_r(z)} \frac{\partial^2}{\partial x_l \partial x_j} \log |z - \xi| \frac{q(\xi)}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi \right| \leq \frac{C_3}{|z|^2 (\log(1/|z|))^2},$$

for some constant  $C_3$  where we have applied Lemma 3.8. On the other hand by using the mean value theorem and the Hölder continuity of  $\kappa$  we find

$$\begin{aligned} &\left| \frac{1}{2\pi} \iint_{K_r(z)} \frac{\partial^2}{\partial x_l \partial x_j} \log |z - \xi| \left( \frac{q(\xi)}{|\xi|^2 (\log(1/|\xi|))^2} - \frac{q(z)}{|z|^2 (\log(1/|z|))^2} \right) d\sigma_\xi \right| \leq \\ &\frac{1}{2\pi} \iint_{K_r(z)} \frac{2}{|z - \xi|^2} \frac{|\kappa(\xi) - \kappa(z)| e^{2w(\xi)}}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi + \frac{1}{2\pi} \iint_{K_r(z)} \frac{2}{|z - \xi|^2} \frac{|\kappa(z)| |e^{2w(\xi)} - e^{2w(z)}|}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi + \\ &\frac{1}{2\pi} \iint_{K_r(z)} \frac{2}{|z - \xi|^2} |q(z)| \left| \frac{1}{|z|^2 (\log(1/|z|))^2} - \frac{1}{|\xi|^2 (\log(1/|\xi|))^2} \right| d\sigma_\xi \leq \frac{C_4}{|z|^2 (\log(1/|z|))^2} \end{aligned}$$

for some  $C_4 > 0$ . Thus  $w_{zz}(z) = w_{z\bar{z}}(z) = w_{\bar{z}z}(z) = O(|z|^{-2}(\log(1/|z|))^{-2})$  for  $z \rightarrow 0$ , as desired.  $\blacksquare$

### 3.5 Proof of Theorem 1.2 and Theorem 1.4

The proof of Theorem 1.2 is a straightforward application of Theorem 1.1 by noting that the metric  $\lambda(z) |dz|$  has a representation of the form

$$\lambda(z) = e^{u(z)} = \begin{cases} |z|^{-\alpha} e^{v(z)} & \text{if } \alpha < 1 \\ \frac{e^{w(z)}}{|z| \log(1/|z|)} & \text{if } \alpha = 1, \end{cases}$$

where  $u(z)$ ,  $v(z)$  and  $w(z)$  have the properties stated in Theorem 1.1 and Theorem 3.4. We note that the statements (a) and (b) of Theorem 1.2 remain valid under the weaker assumption that the curvature  $\kappa$  is only continuous on  $\mathbb{D}$ . This follows from an inspection of the proofs in §3.1–§3.4.

In order to prove Theorem 1.4 we first note that Theorem 1.1 implies that a conformal Riemannian metric  $\lambda(z)|dz|$  on  $\Omega$  with (Hölder continuous) curvature  $\kappa : \Omega \cup \{0\} \rightarrow \mathbb{R}$  with  $\kappa(0) < 0$  is locally complete at  $z = 0$  if and only if  $\log \lambda$  has order  $\alpha = 1$ . In particular, the hyperbolic metric  $\lambda_\Omega(z)|dz|$  has also order  $\alpha = 1$ . Thus Theorem 1.4 follows immediately from Theorem 1.2.

## 4 Appendix: Potential Theory

In this appendix we provide some essentially well-known but non-standard facts from potential theory such as a Poisson–Jensen formula and some differentiability properties of Newton’s Potential that are extensively and repeatedly used in the course of this paper.

### Proposition 4.1 (Poisson–Jensen)

Let  $u$  be a subharmonic function on  $K_r$  such that  $u \in C^2(K_r \setminus \{0\})$ ,  $\Delta u \in L^1(K_r)$  and

$$\lim_{r \rightarrow 0} \frac{\sup_{|z|=r} u(z)}{\log(1/r)} = 0.$$

Then

$$u(z) = h(z) + \frac{1}{2\pi} \iint_{K_r} \log |z - \xi| \Delta u \, d\sigma_\xi, \quad z \in K_r,$$

where  $h$  is a harmonic function on  $K_r$ .

This can be deduced from Theorem 4.5.1 and Exercise 3.7.3 in [27].

### Proposition 4.2 (Newton–Potential)

(a) Let  $r \leq 1$  and  $q : K_r \rightarrow \mathbb{R}$  be a bounded and integrable function. Then for every  $\alpha < 1$  the function

$$\omega : K_r \rightarrow \mathbb{R}, \quad z = x_1 + ix_2 \mapsto \frac{1}{2\pi} \iint_{K_r} \log |z - \xi| \frac{q(\xi)}{|\xi|^{2\alpha}} \, d\sigma_\xi$$

is continuous in  $K_r$ .

Further,  $\omega \in C^1(K_r)$  for  $\alpha < 1/2$  and  $\omega \in C^1(K_r \setminus \{0\})$  for  $1/2 \leq \alpha < 1$ , where

$$\frac{\partial}{\partial x_j} \omega(z) = \frac{1}{2\pi} \iint_{K_r} \frac{\partial}{\partial x_j} \log |z - \xi| \frac{q(\xi)}{|\xi|^{2\alpha}} \, d\sigma_\xi$$

for  $j \in \{1, 2\}$ .

If, in addition,  $q$  is locally Hölder continuous, then  $\omega \in C^2(K_r)$  if  $\alpha \leq 0$  and  $\omega \in C^2(K_r \setminus \{0\})$  if  $0 < \alpha < 1$ , where

$$\begin{aligned} \frac{\partial^2}{\partial x_l \partial x_j} \omega(z) &= \frac{1}{2\pi} \iint_{K_3} \frac{\partial^2}{\partial x_l \partial x_j} \log |z - \xi| \left( \frac{q(\xi)}{|\xi|^{2\alpha}} - \frac{q(z)}{|z|^{2\alpha}} \right) \, d\sigma_\xi \\ &\quad - \frac{1}{2\pi} \frac{q(z)}{|z|^{2\alpha}} \int_{\partial K_3} \frac{\partial}{\partial x_j} \log |z - \xi| n_l(\xi) \, |d\xi|. \end{aligned} \tag{4.1}$$

Here  $(n_1(\xi), n_2(\xi))^T$  is the unit outward normal at the point  $\xi \in \partial K_3$  and  $q$  is extended to vanish outside of  $K_r$ .

(b) Let  $r < 1$  and let  $q : K_r \rightarrow \mathbb{R}$  be a bounded and integrable function in  $K_r$ . Then the function

$$\omega : K_r \rightarrow \mathbb{R}, \quad z = x_1 + ix_2 \mapsto \frac{1}{2\pi} \iint_{K_r} \log |z - \xi| \frac{q(\xi)}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi$$

belongs to  $C^1(K_r \setminus \{0\})$  and

$$\frac{\partial}{\partial x_j} \omega(z) = \frac{1}{2\pi} \iint_{K_r} \frac{\partial}{\partial x_j} \log |z - \xi| \frac{q(\xi)}{|\xi|^2 (\log(1/|\xi|))^2} d\sigma_\xi$$

for  $j \in \{1, 2\}$ .

If, in addition,  $q$  is locally Hölder continuous, then  $\omega \in C^2(K_r \setminus \{0\})$ . Further,

$$\begin{aligned} \frac{\partial^2}{\partial x_l \partial x_j} \omega(z) &= \frac{1}{2\pi} \iint_{K_3} \frac{\partial^2}{\partial x_l \partial x_j} \log |z - \xi| \left( \frac{q(\xi)}{|\xi|^2 (\log(1/|\xi|))^2} - \frac{q(z)}{|z|^2 (\log(1/|z|))^2} \right) d\sigma_\xi \\ &\quad - \frac{1}{2\pi} \frac{q(z)}{|z|^2 (\log(1/|z|))^2} \int_{\partial K_3} \frac{\partial}{\partial x_j} \log |z - \xi| n_l(\xi) |d\xi|, \end{aligned} \quad (4.2)$$

where  $(n_1(\xi), n_2(\xi))^T$  is the unit outward normal at the point  $\xi \in \partial K_3$  and  $q$  is extended to vanish outside of  $K_r$ .

The result is standard for  $\alpha \leq 0$  (see [10, p.54/55]) and the proof for  $\alpha \leq 0$  can be extended to the cases  $0 < \alpha < 1$ .

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